

On the possibility of extinction in a class of Markov processes in economics

Tapan Mitra^a, Santanu Roy^{b,*}

^a Department of Economics, Cornell University, Ithaca, NY 14853, USA

^b Department of Economics, Southern Methodist University, Dallas, TX 75275-0496, USA

Received 4 July 2005; accepted 9 January 2007

Available online 26 April 2007

Abstract

We analyze the possibility of eventual extinction of a replenishable economic asset (natural resource or capital) whose stocks follow a stationary Markov process with zero as an absorbing state. In particular, the stochastic process of stocks is determined by a given sequence of i.i.d. random variables with bounded support and a positive-valued transition function that maps the current level of the stock and the current realization of the random variable to the next period's stock. Such processes arise naturally in stochastic dynamic models of economic growth and exploitation of natural resources. Under a minimal set of assumptions, the paper identifies conditions for almost sure extinction from all initial stocks as well as conditions under which the stocks enter every neighborhood of zero infinitely often almost surely. Our results emphasize the crucial role played by the nature of the transition function under the worst realization of the random shock and clarifies the role of the “average” rate of growth in the context of extinction.

© 2007 Elsevier B.V. All rights reserved.

JEL classification: D90; O11; O41; Q32

Keywords: Stochastic dynamic model; Extinction; Conservation; Markov process; Transition function; Safe standard

1. Introduction

An important condition for sustainability of economic and ecological systems is that stocks of replenishable physical and natural capital are not depleted to zero over time. Extinction is of particular concern when zero is an absorbing state: that is, once the stock of the asset reaches zero, it is no longer available for future use. An important class of such assets consist of biological and

* Corresponding author. Tel.: +1 214 768 2714; fax: +1 214 768 1821.

E-mail address: sroy@smu.edu (S. Roy).

other renewable ecological resources that are irretrievably lost to future generations once they become extinct. Extinction is also a concern in the macroeconomic study of economies that are caught in poverty traps and are not able to sustain their current meagre levels of capital and wealth. An important factor here is the sensitivity of the growth of capital and other assets to random fluctuations (technological shocks, environmental fluctuations) and the fact that persistent adverse shocks can severely deplete capital stocks. It is important to analyze the possibility of extinction in the presence of such uncertainty. In this paper, we consider a class of Markovian models that are widely used in economics and characterize the probability of extinction in terms of the properties of the transition law.

In particular, we consider a replenishable economic asset whose stocks follows a stationary Markov process $\{y_t\}$. The transition law for the Markov process is of the form:

$$y_{t+1} = G(y_t, r_{t+1}),$$

where $\{r_t\}$ is a sequence of i.i.d. random variables with bounded support and G is a time invariant positive-valued transition function that maps the current stock and the realization of the random shock to the next period's stock. Such processes arise naturally in stochastic dynamic models of economic growth and exploitation of natural resources.

We impose very little structure on the transition function G and in particular, we do not require G to be a continuous function or to be monotonic in current stock. This makes our results applicable to a large class of stochastic dynamic models where the transition function for the stock variable (y_t) is determined by the solution to a dynamic optimization problem that is characterized by non-convexities and stock-effects as result of which the optimal policy need not be monotonic or continuous in the stock variable. Such optimization problems arise naturally in models of optimal management of biological resources where the natural growth function is often assumed to be non-concave and the immediate marginal return from harvesting often depends on the stock of the resource. Our results are also applicable to stochastic dynamic models where the investment behavior is generated through interactive behavior among agents (for example, in a market) and it is difficult to ensure continuity or monotonicity of the transition function for the stock variable generated by the dynamic equilibrium.

Under a minimal set of assumptions, we identify conditions for almost sure extinction from all initial stocks (global extinction) as well as conditions under which the stocks enter every neighborhood of zero infinitely often with probability one. The latter implies that there is no "safe standard of conservation". Our results emphasize the crucial role played by the nature of the transition function under the worst realization of the i.i.d. random shock. We also establish a result that relates extinction to a condition on the "average" rate of growth.

The relevant existing literature is rather small. A useful set of conditions on the transition function under which a stochastic process (of the kind considered in our paper) converges to zero with probability one is contained in Athreya (2003); however, the paper imposes continuity and other conditions on the class of admissible transition functions that are much more restrictive than in our framework. Nishimura et al. (2006) consider the special case of multiplicative shock with a smooth density function whose support is the entire positive real line. They provide conditions on the transition function under which the stochastic process converges globally to a degenerate distribution at zero (in the norm topology). In a more general model of stochastic growth, Kamihigashi (2006) uses a similar argument to show that as long as the marginal product at zero is finite, every feasible path converges to zero almost surely if the random shocks are "sufficiently volatile". All of the above mentioned results

in the existing literature are closely related to the condition outlined in Proposition 4 of this paper.

Section 2 outlines the basic framework. To illustrate the scope of application of our results, Section 3 briefly discusses two classes of dynamic models of renewable resource exploitation that fit our framework. Section 4 contains the main propositions characterizing the possibility of extinction.

2. Basic framework

Time is discrete and indexed by $t = 0, 1, \dots$. There is a single state variable which is the stock of an asset, natural resource or capital. The process is defined by a *transition function* that defines the next period's stock as a time-invariant function of the current stock and the current realization of a random shock. We assume that the shocks are independent and identically distributed. In particular, the random shock is represented by a real number in the interval $I = [a, b]$, where $0 < a < b < \infty$. The law governing this shock (initially) is represented by a distribution, μ , with support $[a, b]$. To do this formally, let Ω be the space of all infinite sequences $(\omega_1, \omega_2, \dots)$ where $\omega_t \in [a, b]$ for $t \in \mathbb{N}$. Denote by \mathcal{B} the collection of Borel subsets of $[a, b]$. Let \mathcal{F} be the σ -algebra generated by cylinder sets of the form $\prod_{n=1}^{\infty} A_n$, where $A_n \in \mathcal{B}$ for all $n \in \mathbb{N}$, and $A_n = I$ for all but a finite number of values of n . For each $t \in \mathbb{N}$, denote by \mathcal{F}_t the σ -algebra generated by cylinder sets of the form $\prod_{n=1}^{\infty} A_n$, where $A_n \in \mathcal{B}$ for all $n \in \mathbb{N}$, and $A_n = I$ for all $n \geq t + 1$. Let P be the product measure over \mathcal{F} generated by the probability distribution μ over $[a, b]$. This defines a probability space (Ω, \mathcal{F}, P) . Next, define the projection $r_t(\omega) = \omega_t$ for $t \in \mathbb{N}$. Then $\{r_t\}_1^{\infty}$ is a sequence of independent and identically distributed random variables on (Ω, \mathcal{F}, P) with a common marginal distribution μ .

The stock (state) is represented by a non-negative real number y . Thus, \mathbb{R}_+ is the natural state space for the Markov process of interest and this will be denoted by Y . The *transition function* is a function $G : Y \times I \rightarrow Y$; thus, $G(y, r)$ defines the stock next period, when the current period stock is y , and r is the realization of the random shock.

The following assumptions on G will be maintained throughout the paper:

- (G.1) $G(0, r) = 0$ for all $r \in I$.
- (G.2) $\rho \equiv \lim_{y \rightarrow \infty} [\frac{G(y, b)}{y}]$ exists, and $\rho < 1$.
- (G.3) G is non-decreasing in r .

Assumption (G.1) implies that zero is an absorbing state; it is a natural assumption in studying extinction issues. Assumption (G.2) reflects the fact that there is a maximum sustainable stock beyond which the resource or capital cannot grow. For renewable natural resources, this may reflect the carrying capacity of the ecosystem. Note that (G.2) implies that $G(y, b) < y$ for all y large enough. We define:

$$K = \inf\{z \in Y : G(y, b) < y \text{ for all } y \in (z, \infty)\}. \quad (2.1)$$

Then, $G(y, b) < y$ for all $y \in (K, \infty)$. Assumption (G.3) requires that $G(\cdot, r)$ is weakly ordered according to r which is satisfied for the case of additive and multiplicative shocks. For analysis of cases where (G.3) is violated, see Mirman and Zilcha (1975) and Majumdar et al. (1989).

Associated with G are two functions, $m : Y \rightarrow Y$ and $M : Y \rightarrow Y$, defined by:

$$m(y) = G(y, a) \quad \text{for } y \in Y, \quad M(y) = G(y, b) \quad \text{for } y \in Y. \tag{2.2}$$

Given assumption (G.3), it is legitimate to refer to M as the *best transition function* and to m as the *worst transition function*.

We also impose the following technical condition on the transition function:

(G.4) For any $y > 0$, $\mu\{r \in I : G(y, r) > G(y, a)\} = 1$. Further, for every y^1, y^2 , satisfying $0 < y^1 < y^2 < \infty$,

$$\sup_{y \in [y^1, y^2]} [G(y, r) - G(y, a)] \rightarrow 0 \text{ as } r \downarrow a.$$

Assumption (G.4) imposes two restrictions. First, from any positive stock y , the event that the stock next period is exactly equal to that corresponding to the worst transition $G(y, a)$, has zero probability. This would always be satisfied if $G(y, r)$ is strictly increasing in r for each $y > 0$, and μ is absolutely continuous with positive density on $[a, b]$. Second, it requires that as r decreases to a , $G(y, r)$ converges to $G(y, a)$ *uniformly* in y on any positive closed interval of stocks. If there is an upper semi-continuous function, $g : Y \rightarrow Y$, such that $G(y, r) = rg(y)$ for all $(y, r) \in Y \times I$ (that is, the random shock is multiplicative), then:

$$\sup_{y \in [y^1, y^2]} [G(y, r) - G(y, a)] = (r - a) \left\{ \sup_{y \in [y^1, y^2]} g(y) \right\} \rightarrow 0 \text{ as } r \downarrow a,$$

so that (G.4) is satisfied.

We state an immediate implication of (G.4), which will be useful in our subsequent analysis.

Lemma 1. *Let $0 < p < p' < \infty$ be given, and suppose that:*

$$d \equiv \sup \left\{ \frac{G(y, a)}{y} : y \in [p, p'] \right\} < 1.$$

Then there exist $\lambda \in (0, (b - a))$ and $\theta \in (0, 1)$ such that:

$$G(y, r) < \theta y \text{ for all } r \in [a, a + \lambda] \text{ and all } y \in [p, p'].$$

Proof. We are given that:

$$d = \sup \left\{ \frac{G(y, a)}{y} : y \in [p, p'] \right\} < 1.$$

Choose $\zeta > 0$ and small enough so that $(\zeta/p) < (1 - d)$. Given (G.4), there is $\lambda \in (0, (b - a))$ such that:

$$\sup \{G(y, a + \lambda) - G(y, a) : y \in [p, p']\} < \zeta,$$

so that for all $y \in [p, p']$,

$$\frac{G(y, a + \lambda) - G(y, a)}{y} < \frac{\zeta}{p}. \tag{2.3}$$

Thus, we have for all $y \in [p, p']$,

$$\frac{G(y, a + \lambda)}{y} = \frac{G(y, a + \lambda) - G(y, a)}{y} + \frac{G(y, a)}{y} < \frac{\zeta}{p} + d. \tag{2.4}$$

Using (2.3) and (2.4), we get:

$$\sup \left\{ \left[\frac{G(y, a + \lambda)}{y} \right] : y \in [p, p'] \right\} \leq \frac{\zeta}{p} + d < 1.$$

Now, choosing $\theta \in (\sup\{[G(y, a + \lambda)/y] : y \in [p, p']\}, 1)$, the lemma is proved. \square

Note that we do not require the transition function $G(y, r)$ to be continuous or monotonic in y . Also, observe that we do not put any restriction on the “slope” of the transition function at zero stock (other than requiring it to be non-negative).

The transition function G defines a stochastic process, which is the principal object of our study. Given any initial stock $y > 0$, we will be concerned with the stochastic process $\{y_t(y, \omega)\}$, where $y_t(y, \omega)$ is \mathcal{F}_t measurable for all $t \in \mathbb{N}$, and:

$$y_1(y, \omega) = G(y, \omega_1), \quad y_{t+1}(y, \omega) = G(y_t(y, \omega), \omega_{t+1}) \quad \text{for } t \in \mathbb{N}. \quad (2.5)$$

Finally, we outline a set of definitions of concepts related to extinction. Extinction is defined in a way so as to encompass the event that the stock is reduced to zero in finite time as well as the event that the stocks, while never being actually reduced to zero, become arbitrarily small over time.

Formally, *extinction* is said to occur from an initial stock $y > 0$, given a realization $\omega \in \Omega$, if:

$$\lim_{t \rightarrow \infty} y_t(y, \omega) = 0. \quad (2.6)$$

A natural object of interest is the probability of the set of realizations of the sequence of random shocks for which extinction occurs, starting from a given stock. This leads to the following definition.

Extinction is said to occur from an initial stock $y > 0$ if:

$$P\{\omega \in \Omega : \lim_{t \rightarrow \infty} y_t(y, \omega) = 0\} = 1.$$

It is of interest to know in what scenarios extinction occurs, independent of the initial stock.

Global extinction is said to occur if extinction occurs from every initial stock $y > 0$.

3. Economic examples: renewable resource harvesting under uncertainty

In this section, we outline two dynamic economic models of renewable resource extraction under uncertainty that fits our basic framework. The resource is non-storable and the stock of the resource in period t is denoted by y_t , the amount of resource harvested in period t ($\leq y_t$) is denoted by c_t and given y_0 , resource stocks evolve over time according to the rule

$$y_{t+1} = f(y_t - c_t, r_{t+1}), \quad t \geq 0,$$

where $\{r_t\}$ is a sequence of i.i.d random environmental shocks affecting the natural growth of the resource and f is the biological production function. Assume that $\{r_t\}$ satisfies all of the restrictions outlined in the previous section, that $f(x, r)$ is continuous and strictly increasing in x on \mathbb{R}_+ and that, in addition, f satisfies all of the restrictions imposed on the function G in the previous section. Note that this allows f to be non-concave in x . The evolution of the resource stocks $\{y_t\}$ depends on the nature of harvesting over time.

3.1. Optimally managed resource

First, consider the model where the resource is optimally managed¹ and harvesting is determined as solution to the following stochastic dynamic optimization problem. Given $y_0 > 0$,

$$\max E \sum_{t=0}^{\infty} \delta^t u(c_t, y_t)$$

subject to

$$0 \leq c_{t+1} \leq y_{t+1} = f(y_t - c_t, r_{t+1}), \quad t \geq 0,$$

where $u(c_t, y_t)$ denotes the immediate welfare from harvesting an amount c_t from a stock of y_t and $\delta \in (0, 1)$ is the discount factor. Assume that u is real valued and continuous on $\{(c, y) : 0 \leq c \leq y, y \geq 0\}$.

Observe that this is a generalized one-sector model of optimal economic growth under uncertainty where one allows for non-convexity in the technology² and the preferences and where the utility may depend not only on consumption but also the level of wealth.³

Using standard stochastic dynamic programming arguments, one can show that there exists a stationary optimal policy, though it may not be unique. Let $x(y)$ be an optimal investment policy function, i.e., it is optimal to harvest an amount $y - x(y)$ when the current stock is y . Under the assumptions outlined above, one can ensure that $x(y)$ is a measurable function—but it is not necessarily continuous or monotonic. The transition law for the stochastic process of resource stocks generated by this optimal policy is given by:

$$y_{t+1} = f(x(y_t), r_{t+1}).$$

Define $G(y, r) = f(x(y), r)$. It is easy to check that all of the assumptions on the function G imposed in the previous section are satisfied.

3.2. Open access resource: competitive market

Next, consider a model where the resource is harvested competitively under open access conditions.⁴ There is a continuum of identical firms of finite measure that harvest the resource. As each firm is of measure zero, it is price-taking and its individual harvesting decision has no effect on the future stock of the resource. The unit cost of harvesting for an individual firm at each date t is a constant $z(y_t)$ that potentially depends on the current stock y_t of the resource. Let $D(p)$ denote the market demand function for the harvested resource. Assume that $D(p)$ is strictly positive valued, continuous and strictly decreasing on \mathbb{R}_+ . Further, assume $z(y) > 0$ for all $y > 0$. Then, in equilibrium, the price in each period t is exactly equal to $z(y_t)$ and the total harvest c_t in period t is implicitly defined by the equilibrium condition:

$$c_t = D(z(y_t)).$$

¹ Olson and Roy (2000) and Mitra and Roy (2006) contain analysis of conservation and extinction in stochastic models of optimal resource management.

² Brock and Mirman (1972) analyse a convex model of stochastic optimal growth. See, Majumdar et al. (1989) for a model of optimal stochastic growth with non-convex technology.

³ See, Nyarko and Olson (1991) for a model of stochastic growth with stock-dependent utility.

⁴ See, for example, Mirman and Spulber (1984).

The transition law for resource stocks is therefore given by:

$$y_{t+1} = f(y_t - D(z(y_t)), r_{t+1}).$$

Define $G(y, r) = f(y - D(z(y)), r)$. It is easy to check that all of the assumptions on the function G imposed in the previous section are satisfied. Observe that the function $y - D(z(y))$ is not necessarily continuous or monotonic (even if $z(y)$ is decreasing in y).

4. Conditions for extinction

It is obvious that if for every $y > 0$, $G(y, r) < y$ with probability one then for almost every realization $\omega \in \Omega$, the sequence $\{y_t(y, \omega)\}$ is strictly decreasing and must, in fact, decline to zero so that global extinction necessarily occurs. Our primary interest is to identify conditions weaker than this which still lead to global extinction.

A necessary condition for extinction from any $y > 0$ is that for almost every realization $\omega \in \Omega$, the sequence $\{y_t(y, \omega)\}$ enters any small neighborhood of zero infinitely often, i.e., is not bounded away from zero (though it may not converge to zero). In the stochastic setting of our model, even if the transition function is such that the resource stocks can grow in size under better realizations of the random shock, runs of bad shocks can reduce the stocks to levels close to zero infinitely often. One might speculate that the probability with which this occurs depends on the nature of the transition function during “bad” realizations of the shock and the way it compares to the transition function during “good” realizations of the shock as well as the relative weight of these realizations. The surprising result is that the condition for stocks to be infinitesimally close to zero infinitely often with probability one depends only on the worst transition function m and the behavior of the resource stocks under better realizations of the random shock is irrelevant.

Proposition 2. *Suppose that*

$$\sup \left\{ \left[\frac{m(y)}{y} \right] : y \geq h \right\} < 1 \quad \forall h > 0. \tag{4.1}$$

Then, for every $y > 0$,

$$P \left\{ \omega \in \Omega : \liminf_{t \rightarrow \infty} y_t(y, \omega) = 0 \right\} = 1.$$

Proof. To begin, consider a situation where $K = 0$. Then $M(\hat{y}) < \hat{y}$, $\forall \hat{y} > 0$. Let $\{\bar{y}_t(y)\}$ be the sequence of non-negative real numbers defined by

$$\bar{y}_0(y) = y, \bar{y}_{t+1}(y) = M(\bar{y}_t(y)).$$

It is easy to check that $\{\bar{y}_t\} \downarrow 0$ and that

$$P\{\omega \in \Omega : y_t(y, \omega) \leq \bar{y}_t \forall t \geq 0\} = 1.$$

It follows that

$$P \left\{ \omega \in \Omega : \lim_{t \rightarrow \infty} y_t(y, \omega) = 0 \right\} = 1.$$

In the rest of the proof, confine attention to the case where $K > 0$. We want to show that:

$$P \left\{ \omega \in \Omega : \liminf_{t \geq 0} y_t(y, \omega) > 0 \right\} = 0 \quad \forall y > 0.$$

Under (G.2), it is sufficient to show this for all $y \in (0, K]$. Suppose that there is some $y \in (0, K]$, such that:

$$P \left\{ \omega \in \Omega : \liminf_{t \geq 0} y_t(y, \omega) > 0 \right\} > 0. \tag{4.2}$$

Then, there is some $s > 0$ such that:

$$P \left\{ \omega \in \Omega : \liminf_{t \geq 0} y_t(y, \omega) > s \right\} > 0. \tag{4.3}$$

Observe that:

$$\begin{aligned} & \{ \omega \in \Omega : \liminf_{t \geq 0} y_t(y, \omega) > s \} \\ &= \{ \omega \in \Omega : \text{there exists } T(\omega) < \infty \text{ such that } y_t(y, \omega) > s \forall t > T(\omega) \} \\ &\subset \bigcup_{T=0}^{\infty} \{ \omega \in \Omega : y_t(y, \omega) > s \forall t > T \}. \end{aligned}$$

Thus, we have:

$$P\{ \omega \in \Omega : \liminf_{t \geq 0} y_t(y, \omega) > s \} \leq \sum_{T=0}^{\infty} P\{ \omega \in \Omega : y_t(y, \omega) > s \forall t > T \}. \tag{4.4}$$

We now need to estimate the probabilities appearing in the sum in the inequality (4.4). By Lemma 1, there are $\lambda \in (0, (b - a))$ and $\theta \in (0, 1)$, such that:

$$G(y, r) < \theta y \forall r \in [a, a + \lambda] \forall y \in [s, K]. \tag{4.5}$$

Let N be the smallest positive integer such that $\theta^N K < s$. Let A be the event defined by:

$$A = \{ \omega \in \Omega : \exists t \geq T \text{ such that } \omega_{t+i} \in [a, a + \lambda] \text{ for } i = 1, \dots, N \}.$$

Observe that for $\omega \in A$, $y_{t+N}(y, \omega) < \theta^N y_t(y, \omega)$ for some $t \geq T$ and since $y_t(y, \omega) \leq K$ for almost every $\omega \in \Omega$, we have that for almost every $\omega \in A$, $y_{t+N}(y, \omega) < \theta^N K < s$ for some $t \geq T$, so that:

$$\begin{aligned} & P\{ \omega \in \Omega : \exists t \geq T \text{ such that } y_t(y, \omega) \leq s \} \\ &\geq P\{ \omega \in \Omega : \exists t \geq T \text{ such that } \omega_{t+i} \in [a, a + \lambda] \forall i = 1, \dots, N \} = 1, \end{aligned} \tag{4.6}$$

the last equality in (4.6) following from the fact that $\{r_i\}$ is a sequence of i.i.d. random variables with marginal distribution μ and the fact that the support of this distribution is the interval $[a, b]$ so that $\mu\{r \in [a, a + \lambda]\} > 0$. Thus, we have:

$$P\{ \omega \in \Omega : y_t(y, \omega) > s \forall t \geq T \} = 0 \text{ for each } T \geq 0,$$

which implies that the right-hand side of (4.4) is zero. This contradicts (4.3) and, therefore, (4.2), concluding the proof. \square

Proposition 2 provides a condition under which no matter how large the initial stock, the stochastic process enters every neighborhood of zero infinitely often with probability one. This is, however, consistent with the stochastic process converging in distribution to a limit distribution whose support includes zero but which does not assign strictly positive mass to zero. In the bio-economics literature, there is a concept of a *safe standard of conservation* —for any initial stock

higher than the safe standard, all future stocks lie above the safe standard. While the condition in Proposition 2 does not ensure that stocks converge to zero, it does imply the non-existence of any such “safe standard” for almost every realized sample path.

It should be noted here that the proof of Proposition 2 uses the fact that the probability that the random shock lies in any neighborhood of the “worst realization” a is strictly positive; the latter follows from our assumption that the support of the common distribution of the i.i.d. random shocks is the interval $[a, b]$.⁵

Our next result shows that if there is a neighborhood of zero, however small, where stocks decline even under the best realization of the random shock, then the condition in Proposition 2 is sufficient to ensure global extinction.

Proposition 3. *Suppose that (4.1) holds and further, there exists $\alpha > 0$ such that $M(y) < y$ for all $y \in (0, \alpha)$. Then, global extinction occurs, i.e.,*

$$P\{\omega \in \Omega : \lim_{t \rightarrow \infty} y_t(y, \omega) = 0\} = 1 \quad \forall y > 0.$$

Further, for any $\varepsilon > 0$ and $\tau(y, \omega) = \inf\{t \geq 0 : y_t(y, \omega) \leq \varepsilon\}$, $E(\tau(y, \omega)) < \infty$, for all $y > 0$.

Proof. To begin, consider the situation where $K = 0$. Fix any $y > 0$. Let $\{\bar{y}_t(y)\}$ be the sequence of non-negative real numbers defined by

$$\bar{y}_0(y) = y, \bar{y}_{t+1}(y) = M(y_t), \quad t \geq 0.$$

and following the arguments made at the beginning of Proposition 2, observe that $\{\bar{y}_t(y)\} \downarrow 0$ and

$$P\{\omega \in \Omega : y_t(y, \omega) \leq \bar{y}_t(y) \forall t \geq 0\} = 1,$$

so that

$$P\{\omega \in \Omega : \lim_{t \rightarrow \infty} y_t(y, \omega) = 0\} = 1.$$

Further, since $\{\bar{y}_t(y)\} \downarrow 0$, for any $\varepsilon > 0$, there exists $\tau(y) < \infty$ such that $\bar{y}_t(y) \leq \varepsilon, \forall t \geq \tau(y)$, which implies that

$$P\{\omega \in \Omega : y_t(y, \omega) \leq \varepsilon \forall t \geq \tau(y)\} = 1.$$

This, in turn, implies that $\tau(y, \omega) = \inf\{t \geq 0 : y_t(y, \omega) \leq \varepsilon\} \leq \tau(y)$ almost surely so that, in particular, $E(\tau(y, \omega)) \leq \tau(y) < \infty$. In the rest of the proof, we confine attention to the situation where $K > 0$. First, we show that global extinction occurs. Choose ϑ in $(0, \alpha)$. It is obvious that extinction occurs from any $y \in (0, \vartheta)$. It is sufficient to show the result for $y \in (\vartheta, K]$, where K is defined in (2.1). Pick any $\bar{y} \in (\vartheta, K]$. Since (4.1) holds, using Proposition 2, we know that $\liminf_{t \rightarrow \infty} y_t(y, \omega) = 0$ almost surely and hence, we have:

$$P\{\omega \in \Omega : \exists t \text{ for which } y_t(\bar{y}, \omega) \leq \vartheta\} = 1.$$

Also, if $y_t(\bar{y}, \omega) \leq \vartheta$, then $y_{t+1}(\bar{y}, \omega) = G(y_t(\bar{y}, \omega), \omega_{t+1}) \leq G(y_t(\bar{y}, \omega), b) = M(y_t(\bar{y}, \omega)) < y_t(\bar{y}, \omega)$, and so $\{y_t(\bar{y}, \omega)\}$ must converge to zero. Thus, we have:

$$P\left\{\omega \in \Omega : \lim_{t \rightarrow \infty} y_t(\bar{y}, \omega) = 0\right\} = P\left\{\omega \in \Omega : \liminf_{t \rightarrow \infty} y_t(\bar{y}, \omega) = 0\right\} = 1.$$

⁵ An alternative would be to require that the random shocks assign strictly positive probability mass to the “worst realization” a .

Next, we show that for any $\varepsilon > 0$ and $\tau(y, \omega) = \inf\{t \geq 0 : y_t(y, \omega) \leq \varepsilon\}$, $E(\tau(y, \omega)) < \infty$ for all $y > 0$. Without loss of generality, suppose that $\varepsilon < \alpha$. Since (4.1) holds, we can use Lemma 1 to obtain $\lambda \in (0, (b - a))$ and $\theta \in (0, 1)$, such that:

$$G(y, r) < \theta y \text{ for all } r \in [a, a + \lambda] \text{ and all } y \in [\varepsilon, K].$$

Let N be the smallest positive integer such that $\theta^N K < \varepsilon$. For each $t \geq 0$, let B_t be the event defined by:

$$B_t = \{\omega \in \Omega : \omega_{t+i} \in [a, a + \lambda] \text{ for } i = 1, \dots, N\}.$$

Observe that for each $t \geq 0$, for $\omega \in B_t$, we have $y_{t+N}(y, \omega) \leq \theta^N y_t(y, \omega) \leq \theta^N K < \varepsilon$. Let $q = \mu\{r \in [a, a + \lambda]\}$; then, $0 < q < 1$. Now, for any $t \geq 0$, we have:

$$\begin{aligned} P\{\omega \in \Omega : y_{t+N}(y, \omega) < \varepsilon\} &\geq P\{\omega \in \Omega : y_t(y, \omega) < \varepsilon\} + P\{\omega \in \Omega : y_t(y, \omega) \geq \varepsilon\}P(B_t) \\ &= P\{\omega \in \Omega : y_t(y, \omega) < \varepsilon\} + P\{\omega \in \Omega : y_t(y, \omega) \geq \varepsilon\}q^N, \end{aligned}$$

so that:

$$P\{\omega \in \Omega : y_{t+N}(y, \omega) \geq \varepsilon\} \leq P\{\omega \in \Omega : y_t(y, \omega) \geq \varepsilon\}(1 - q^N). \tag{4.7}$$

Define for $t \geq 0$, $A_t = \{\omega \in \Omega : y_t(y, \omega) \geq \varepsilon\}$. Denote $(1 - q^N)$ by v ; so $v \in (0, 1)$. Then, for any $k \geq 0$ and $j \geq 1$, (4.7) implies that:

$$\frac{P(A_{N(k+1)+j})}{P(A_{Nk+j})} \leq v. \tag{4.8}$$

Since $\tau(y, \omega)$ is a positive integer valued random variable, we have:

$$E(\tau(y, \omega)) = \sum_{t=1}^{\infty} P(A_t), \tag{4.9}$$

by using the corollary in Chung (1974, p. 43). For $j = 1, \dots, N$, define:

$$C_j = \sum_{k=0}^{\infty} P(A_{j+kN}). \tag{4.10}$$

It follows from (4.8) that $C_j < \infty$ for each $j \in \{1, \dots, N\}$. Using (4.9) and (4.10), we get:

$$E(\tau(y, \omega)) = \sum_{t=1}^{\infty} P(A_t) = \sum_{j=1}^N C_j < \infty,$$

which completes our proof. \square

Proposition 3 characterizes a class of transition functions for which stocks converge to zero with probability one. It also ensures two other properties: (i) for almost every sample path, stocks eventually monotonically decrease to zero and (ii) stocks fall below any small positive threshold in finite expected time. The conditions required for these to occur are that

- (a) in a neighborhood of zero, however small, the stock does not grow even under the best realization of the random shock and
- (b) that the stock always declines under the worst realization of the random shock.

In the renewable natural resource literature, (a) is recognized as a property of the growth function of biological species that are characterized by “critical depensation” (see, Clark, 1990). It also arises in some models of physical capital accumulation and stochastic growth when the production function is characterized by strong increasing returns to scale for small capital stocks so that the technology does not allow the economy to be sustainable when it is very poor. Proposition 3 shows that in all such situations, if (b) holds, extinction occurs from all initial conditions with probability one and the growth rates under better realizations of the random shocks are not relevant to the eventual destiny of the stocks. Of course, the existence of a maximum sustainable stock plays an important role in this result. The general idea behind this result was first exploited by Majumdar and Radner (1992) in their analysis of survival under production uncertainty with a minimum consumption constraint. However, condition (4.1) is more general. Proposition 3 serves to emphasize the need to understand and focus on the performance of ecological and economic processes under the worst possible circumstances.

We provide below an alternative condition which ensures that stocks converge to zero with probability one under a condition that puts a bound on an “average” of the growth rate implied by the transition function.

For every $r \in [a, b]$, let $\lambda(r) = \sup\{(G(y, r))/y : y > 0\}$. As long as transition function has a finite slope at zero even under the best realization of the random shock, $\lambda(r)$ is a non-negative real valued function.

Proposition 4. Assume that (i) $\lambda(a) > 0$ and $\lambda(b) < +\infty$ and (ii) $E[\ln(\lambda(r))] < 0$. Then, global extinction occurs, i.e.,

$$P \left\{ \omega \in \Omega : \lim_{t \rightarrow \infty} y_t(y, \omega) = 0 \right\} = 1 \quad \forall y > 0.$$

Proof. Consider any initial stock $y > 0$. Let

$$C = \{ \omega \in \Omega : \exists \tau(\omega) \geq 0, y_t(y, \omega) = 0 \forall t \geq \tau(\omega) \}$$

By definition, $\omega \in C$ implies that $y_t(y, \omega) \rightarrow 0$. It is sufficient to show that $y_t(y, \omega) \rightarrow 0$ almost surely on $\Omega - C$. In what follows, confine attention to $\omega \in \Omega - C$ so that $y_t(y, \omega) > 0, \forall t \geq 0$. Observe that

$$\begin{aligned} y_{t+1}(y, \omega) &= G(y_t(y, \omega), r_{t+1}(\omega)) \\ &= \left\{ \frac{G(y_t(y, \omega), r_{t+1}(\omega))}{y_t(y, \omega)} \right\} y_t(y, \omega) \leq \lambda(r_{t+1}(\omega)) y_t(y, \omega), \end{aligned}$$

so that by iteration, we get:

$$y_{t+1}(y, \omega) \leq \left\{ \prod_{j=1}^{t+1} \lambda(r_j(\omega)) \right\} y.$$

Taking log on both sides, and time-averaging, we obtain:

$$\frac{1}{t+1} \ln y_{t+1}(y, \omega) \leq \left\{ \frac{1}{t+1} \sum_{j=1}^{t+1} \ln[\lambda(r_j(\omega))] \right\} + \frac{1}{t+1} \ln y. \tag{4.11}$$

Since $\{r_t(\omega)\}$ are i.i.d. random variables, and $\zeta \equiv E[\ln(\lambda(r))]$ < 0, by the strong law of large numbers, we have:

$$P \left\{ \omega \in \Omega : \lim_{t \rightarrow \infty} \left\{ \frac{1}{t+1} \sum_{j=1}^{t+1} \ln[\lambda(r_j(\omega))] \right\} = \zeta \right\} = 1. \tag{4.12}$$

Note that $[1/(t + 1)] \ln y \rightarrow 0$ as $t \rightarrow \infty$. Thus, using (4.11) and (4.12), we get:

$$P\{\omega \in \Omega - C : \text{there exists } T(\omega) < +\infty, \text{ such that } \left[\frac{1}{t+1} \right] \ln y_{t+1}(y, \omega) < \left(\frac{\zeta}{2} \right) \text{ for all } t \geq T(\omega)\} = P(\Omega - C). \tag{4.13}$$

This implies:

$$P \left\{ \omega \in \Omega - C : \lim_{t \rightarrow \infty} [\ln y_{t+1}(y, \omega)] = -\infty \right\} = P(\Omega - C),$$

which can be rewritten as:

$$P \left\{ \omega \in \Omega - C : \lim_{t \rightarrow \infty} [y_t(y, \omega)] = 0 \right\} = P(\Omega - C).$$

This concludes the proof. \square

Athreya (2003) establishes a sufficient condition for convergence to zero of a Markov process of the kind used in Proposition 4 above. In our context, his condition reduces to the requirements that (a) for each $r \in I$, $\lim_{y \downarrow 0} [G(y, r)/y]$ exists and is positive and finite, (b) for each $r \in I$, $\lim_{y \downarrow 0} [G(y, r)/y] > [G(y, r)/y]$ for all $y > 0$ and (c) $E \ln[\lim_{y \downarrow 0} \{G(y, r)/y\}] < 0$. It is easy to check that under these conditions, $\lambda(a) > 0$ and $\lambda(b) < +\infty$ and $E[\ln(\lambda(r))] < 0$ so that our conditions in Proposition 4 are satisfied. Observe that requirement (b) in Athreya’s paper is not satisfied if, for example, $G(y, r)$ is S-shaped in y for each r ; our conditions do allow for S-shaped transition functions. The method of proof followed in Athreya (2003) goes through under our weaker conditions.

Nishimura et al. (2006) establish a result similar to Proposition 4 in a one-sector stochastic growth model with multiplicative shocks where the transition function is monotonic and where the common marginal distribution of the random shocks has a smooth density with support equal to \mathbb{R}_+ (so that from any current stock one reaches any interval of stocks, however high or low, with strictly positive probability).

In a more general one-sector stochastic growth model with multiplicative shocks, Kamihigashi (2006) uses the same argument as in Proposition 4 to show that every feasible path converges to zero with probability one if shocks are sufficiently volatile and the marginal product at zero is finite. His framework allows for shocks that are not necessarily i.i.d and for transition functions that are not necessarily monotonic or continuous in the stock variable.⁶

⁶ Proposition 4 was originally included in Mitra and Roy (2003), a working paper. After writing that paper, we became aware of the results by Kamihigashi and Nishimura et al that parallel Proposition 4.

References

- Athreya, K.B., 2003. Stationary measures for some Markov chain models in ecology and economics. *Economic Theory* 23, 107–122.
- Brock, W.A., Mirman, L.J., 1972. Optimal economic growth and uncertainty: the discounted case. *Journal of Economic Theory* 4, 479–513.
- Chung, K.L., 1974. *A Course in Probability Theory*, 2nd ed. Academic Press, New York.
- Clark, C.W., 1990. *Mathematical Bioeconomics: The Optimal Management of Renewable Resources*, 2nd ed. John Wiley & Sons.
- Kamihigashi, T., 2006. Almost sure convergence to zero in stochastic growth models. *Economic Theory* 29, 231–237.
- Majumdar, M., Mitra, T., Nyarko, Y., 1989. Dynamic optimization under uncertainty: nonconvex feasible set. In: Feiwel, G. (Ed.), *Joan Robinson and Modern Economic Theory*. Macmillan, pp. 545–590.
- Majumdar, M., Radner, R., 1992. Survival under production uncertainty. In: Majumdar, M. (Ed.), *Equilibrium and Dynamics*. Macmillan, pp. 179–200.
- Mirman, L.J., Spulber, D., 1984. Uncertainty and markets for renewable resources. *Journal of Economic Dynamics and Control* 8, 239–264.
- Mirman, L.J., Zilcha, I., 1975. On optimal growth under uncertainty. *Journal of Economic Theory* 11, 329–339.
- Mitra, T., Roy, S., 2003. Optimal exploitation of renewable resources under uncertainty and the extinction of species. Working Paper 03–10. Center for Analytic Economics, Cornell University, Ithaca, NY.
- Mitra, T., Roy, S., 2006. Optimal exploitation of renewable resources under uncertainty and the extinction of species. *Economic Theory* 28, 1–23.
- Nishimura, K., Rudnicki, R., Stachurski, J., 2006. Stochastic optimal growth with nonconvexities. *Journal of Mathematical Economics* 42, 74–92.
- Nyarko, Y., Olson, L.J., 1991. Stochastic dynamic models with stock-dependent rewards. *Journal of Economic Theory* 55, 161–168.
- Olson, L.J., Roy, S., 2000. Dynamic efficiency of conservation of renewable resources under uncertainty. *Journal of Economic Theory* 95, 186–214.